0.1 Introduction to Complex Analysis

- Talk on elementary complex analysis by a high school freshman
- What is complex analysis? Study of functions of complex numbers
- Why is this relevant? Zeta function, quantum mechanics, electrical and nuclear engineering
- Proving some fundamental results and theorems today, working through the actual math tsx

0.2 Defining Terms

NOTATION

- Assume viewer knows complex numbers and single-variable calculus, multivariable is useful
- z = complex number
- Complex plane - real axis, imaginary axis
- $\frac{df}{dx}$ - dee eff dee ecks, derivative of 1-var function $f(x)$ with respect to x ofc
- New to multivariable - $\frac{\partial f}{\partial x}$, del eff del ecks, partial derivative of f with respect to x in a multi-var function $f(x, y, ...)$ where the other vars are constant
- $\iint_D f(x)dx$ - area integral over a region D, volume under a surface instead of area under a curve
- $\oint_C f(z)\, dz$ - integral over a closed curve C, ccw + cw - somex no circle
- line/contour integrals and compdiff follow similar rules to real vers

TERMS

- Complex differentiable; similar to real differentiability, that $lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists at z_0
- $f(z)$ is holomorphic over a set if it is compdiff at every point of that set
- $f(x)$ is analytic if expressable as convergent power series: $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. Proof that all holomorphic functions are analytic (relate back to ζ)
- mention that some things (eg del cont) will be skipped even when necessary preconditions

0.3 Deriving Cauchy-Riemann

- comp to real diff as follows: holomorphic $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ for two real realdiff 2-var u, v
- $f(z)$ is guaranteed to be compdiff from any dir (holomorphic), incl from vertical/horiz dirs
- $\Delta z = \Delta x + i\Delta y$: horizontal, $\Delta y = 0$ so $\Delta z = \Delta x$ then:

$$\begin{aligned} f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x+iy) - f(x+iy)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(u(x+\Delta x, y) + iv(x+\Delta x, y)) - (u(x,y) + iv(x,y))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned} \tag{1}$$

- same for vertical so $\Delta x = 0$ so $\Delta z = i\,\Delta y$ (rmb $\frac{1}{i} = -i$, factor first, cancel second)
- then $f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ meaning $\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ and since Re/Im have to equal:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{2}$$

0.4 Gen. Stokes'/Green's Theorems and Deriving the Integral Theorem

- Start at MVC over reals: Generalized Stokes equation, somex FTMVC: $\int_{\partial\Omega} \omega = \int_\Omega d\omega$ - integral over boundary region big omega to integral over region itself
- 2D complex plane: Green's theorem spec. case: for 2 2-var real realdiff funcs $L(x,y); M(x,y)$:

$$\oint_C (Ldx + Mdy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy \tag{3}$$

- Looks like Cauchy-Riemann! Again, holomorphic $f(z)$ separated into Re/Im: $u(x,y) + iv(x,y)$. Take integral over γ of $f(z)\,dz$, also separate differential: $dz = dx + i\,dy$:

$$\oint_\gamma f(z)\,dz = \oint_\gamma (u+iv)(dx + i\,dy) \to \oint_\gamma (u\,dx - v\,dy) + i\oint_\gamma (v\,dx + u\,dy) \tag{4}$$

- Expand and factor. Green's theorem: replace contour integrals with area integrals over D:

$$\oint_\gamma f(z)\,dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \tag{5}$$

- $f(z)$ is holomorphic over D, must be compdiff at $z \in D$, ergo Cauchy-Riemann must be satisfied by u, v. Substitute:

$$\oint_\gamma f(z)\,dz = \iint_D \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy = \iint_D 0\,dx\,dy + \iint_D 0\,dx\,dy = 0 \tag{6}$$

- $\oint_\gamma f(z)\,dz = 0$ for holomorphic $f(z)$ over D bounded by γ. Called Cauchy's integral theorem, one of the major fund. results in compan. Can be proven only by using the fact that $f(z)$ is compdiff over the region containing γ - eg holomorphic!

0.5 Deriving Cauchy's Integral Formula

- State the formula, explain derivation: val of complex func at pt if you know values around it & func is holomorphic

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \tag{7}$$

- Start with $f(z)$ holomorphic over open set D, and closed ccw circle C in D with z_0 within C and an arbitrarily small circle γ centered at z_0 entirely in C.
- "So wait a minute, doesn't $\oint_C \frac{f(z)}{z-z_0} dz$ evaluate to 0 by Cauchy's integral theorem (same for γ)?" No, since within C and γ at point z_0 $\frac{f(z)}{z-z_0}$ is undefined, eg not holomorphic at z_0
- To fix, combine C with γ to form two ccw loops: C^+ and C^-. Neither contains z_0, eg $\frac{f(z)}{z-z_0}$ is holomorphic for each, and the integral of the function over each is 0
- We can see that the sum of these two integrals is equal to the difference of the integral over C and γ: the outer loop around C is ccw, the inner loop around γ is cw, and the lines cancel:

$$\oint_C \frac{f(z)}{z - z_0} dz - \oint_\gamma \frac{f(z)}{z - z_0} dz = \oint_{C^+} \frac{f(z)}{z - z_0} dz + \oint_{C^-} \frac{f(z)}{z - z_0} dz \tag{8}$$

- The right side equals zero, implying the integrals over C and γ are equal. We can then rewrite the right side in terms of $f(z_0)$:

$$\oint_\gamma \frac{f(z)}{z - z_0} dz = \oint_\gamma \frac{f(z) + f(z_0) - f(z_0)}{z - z_0} dz = \oint_\gamma \frac{f(z_0)}{z - z_0} + \oint_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz \tag{9}$$

- Treat these like integrals you're used to: separate addition, separate out $f(z_0)$ b/c independent of the var of differentiation

$$\oint_\gamma \frac{f(z_0)}{z - z_0} + \oint_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz = f(z_0) \oint_\gamma \frac{1}{z - z_0} dz + \oint_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz \tag{10}$$

- It can be proven using parameterization and contour deformation that term $\oint_\gamma \frac{1}{z-z_0} dz = 2\pi i$ for any closed curve γ (using $z_0 + x_0 + e^{it}$ as the circle around the origin of radius 1)

$$\oint_\gamma \frac{1}{z - z_0} dz = \int_0^{2\pi} \frac{i\, e^{it}\, dt}{e^{it}} = \int_0^{2\pi} i\, dt = 2\pi i \tag{11}$$

- We can then prove $\oint_\gamma \frac{f(z)-f(z_0)}{z-z_0} dz = 0$: this function is necessarily holomorphic because $f(z)$ is (there's a bit more to it but interest of time), and here we can apply Cauchy's theorem to let this equal 0. We get our intended original result.

0.6 Proof of Analyticity

- Take an open ball A contained in D centered at a, rewrite the integral formula as:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)(1-\frac{z_0-a}{z-a})} dz = \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \frac{f(z)}{z-a} \left(\frac{z_0-a}{z-a} \right)^n dz \tag{12}$$

- $|\frac{z_0-a}{z-a}| < 1$ guaranteed since $|z_0 - a| < |z - a|$ since z_0 is within D and z is on C, and an open ball is a circle; then move the other terms inside the infinite sum
- taking each term individ.: $f(z)$ is continuous and therefore bounded by some finite value M; $|\frac{1}{z-a}|$ is also finite and equal to r (radius); $\left(\frac{z_0-a}{z-a}\right)^n < 1$ and therefore for some $0 \le N < 1$ $|\left(\frac{z_0-a}{z-a}\right)^n| < N$
- therefore $|\frac{1}{t-a}||\frac{(z_0-a)^n}{(z-a)^n}||f(t)| \le \frac{1}{r} MN^n$ and the infinite sum of that sequence converges, so by Weierstrass M the inf sum converges uniformly and absolutely on D, meaning we can swap the sum and integral:

$$f(z_0) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_C \frac{f(z)}{z-a} \left(\frac{z_0-a}{z-a} \right)^n dz \tag{13}$$

- factor out the term independent of the var of int and you get a power series expansion!

$$\sum_{n=0}^{\infty} (z_0-a)^n \left(\frac{1}{2\pi i} \oint_C \frac{f(z)}{(t-a)^{n+1}} dz \right) = \sum_{n=0}^{\infty} c_n (z_0-a)^n \tag{14}$$

- rmb this only holds for open balls - contour deformation allows it to work for everything as long as it's not around another singularity
- corollary - radius of convergence is equivalent to the distance to nearest singularity b/c greatest radius of open ball within which $f(z)$ is holomorphic

0.7 Consequences and Conclusion

- Things like complex rigidity, analytic continuation, convergent power series

0.8 Intro

- introduce yourself
- introduce compan
- relevant - zeta function (distribution of the primes), quantum mechanics (eg wave function), electrical/nuclear engineering (reactor kinetics, plasma physics)
- walking through some fundamental proofs, but i'm not very good and do skip some things (intentionally or not) so wikipedia and other math sources are a good bet

0.9 Conclusion

- some corollaries of the last proof
- radius of convergence equals distance to nearest singularity
- identity theorem and complex rigidity - knowing the values on a finite area tells you the values everywhere
- holomorphic are inf differentiable since power series are - formula derived from Cauchy's int formula also allows us to calc derivatives at every point - specifically the nth derivative at z_0 is:

$$\frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \tag{15}$$

- these basics plus another fundamental result from the integral formula that i didn't mention in the interest of time and simplicity, the residue theorem, form the basis of most complex analysis
- that's all i have time for without going over an hour (this talk probably has anyway), but there's a lot more to complex analysis than this - wikipedia is a good source if i didn't explain something well enough, which probably happened, and there are plenty of other sources online. the comments section also exists, as does chatgpt, if you have questions. thanks for listening!