Practice with Complex Numbers

If you are not already familiar with this topic, I'd urge you to look at Chapter 1 of Laforest's textbook. The following short set of questions should test your understanding of complex numbers.

1 Basic Operations

Reading: Sections 1.1, 1.2 of Laforest

1.1 Some Calculations

Let $a = 2 + 3i$ and $b = 3 - 4i$. Simplify the following expressions, by expressing them in the form $z = x + iy$, for real numbers x, y.

- $z = a + 3b$
- $z = 1 + ab$
- $z = \overline{a + b}$
- $z = |a + b|$
- (**Extra Credit**) $z = \frac{a}{b} - \frac{b}{a}$
  Hint: For this last part, Example 1.10 from Laforest may be helpful

1.2 Complex Norm

Prove that for arbitrary complex number z, $|z|^2 = z\bar{z}$.

2 Euler's formula (Extra Credit)

Reading: Appendices A.3 and A.4, then Section 1.3 of Laforest

2.0.1 Part (a)

Evaluate the following exponents:

- $e^{i\pi}$

- $e^{i\frac{\pi}{2}}$
- $e^{-i\frac{\pi}{6}}$

2.0.2 Part (b)

Show that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{1}$$

What would be the analogous expression for $\sin x$?

2.0.3 Part (c)

Write $z = 1 - \sqrt{3}\,i$ in polar form, then evaluate z^6.

2.0.4 Part (d)

Try to prove the sine and cosine addition formula using Euler's formula. You need to show

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

Hint: Consider $z_1 = e^{ix}$ and $z_2 = e^{iy}$. What does the Euler formula tell us $z_1 z_2$ is equal to?