Postulates of Quantum Mechanics

The goal of this assignment is to familiarize you with the basics of quantum mechanics.

1 Measurement Example

Reference: Sections 3.1 - 3.4 of Laforest

The "computational basis" of qubit states is given by vectors $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Another basis is given by $|\alpha\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$, $\beta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

1.1 Part (a)

Check that both bases are *orthonormal* (their basis vectors are orthogonal to each other and have a norm of 1)

1.2 Part (b)

Say our qubit was initially in a state $|0\rangle$ and was measured in the basis of $|\alpha\rangle$ and $|\beta\rangle$. What will the state of the qubit be immediately following the measurement? What are the probabilities of the outcomes?

1.3 Part (c)

Following the first measurement, the qubit is measured again, this time in the basis of $|0\rangle$, $|1\rangle$. What will be the probabilities of the two outcomes?

*Side note: One possible physical use of this problem has to do with measurements of electron **spin**. The Stern-Gerlach experiment showing that the spin measurements result in discrete values is one of the experiments that originally lead to quantum mechanics. You can see a cool animation explaining this experiment here, and check out an online simulator here. In our problem, we can take $|0\rangle$ and $|1\rangle$ to be the two possible outcomes of the spin measurement in the z - direction, and $|\alpha\rangle$ and $|\beta\rangle$ to be the two possible outcomes of the spin measurement in the x - direction.*

2 Distinguishing Between States

Two boxes each produce a stream of qubits. Box A produces the qubits all in the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$. Box B randomly produces qubits in states $|0\rangle$ and $|1\rangle$, each with probability 1/2. We have one of the boxes, but it is unmarked and so we do not know which kind it is. Describe an experiment on the qubits that can tell the difference between box A and box B. Can you reliably tell the difference between the boxes by examining only one of the qubits?

3 (Extra Credit) Mach-Zehnder Interferometer

Reference: Section 2.1 of Schumacher and Westmoreland. You can also check out the simulator of this experiment here.

Consider the experimental setup from Figure 2.6 (also described in a slightly different way in Figure 2.1) of Schumacher and Westmoreland.

- As a warmup, say the "phase shifter" item in Figure 2.6 (grey block labelled by "ϕ") is absent. Let's say that, like in the Figure 2.6, there is a single photon entering the setup from below, which according to the notation from the textbook, can be described by a state vector $|v\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. What are the probabilities that the two detectors D_0 and D_1 register the photon? *Hint: the action of the left beamsplitter (BS1) on the state of photon can be described by a matrix* $B_l = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$*, while the action of the right beamsplitter (BS2) can be described by the matrix* $B_u = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$.
- Now, consider the same problem but with the "phase shifter" present in the lower beam. The setting is such that the phase of the lower beam component is shifted by π (equivalently, we may say that the lower beam component is multiplied by -1). Its effect on the state of the photon is given by the matrix $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. What will be the probabilities of a single photon initially described by the quantum state $|v\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ being found in each of the two detectors?
- Now, say we have a phase shift by an arbitrary phase ϕ, described by the matrix $P = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$. Find the probabilities of the photon being found in each of the two detectors, and show that they add up to 1.
- Now, say the phase shifter is gone but the photon enters the interferometer in a superposition $|v\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. What are the probabilities of our photon being found in each of the two detectors?