Practice with Complex Numbers: Homework

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Yes, I typeset this myself in LaTeX- the assignment took about 2 hours. Please let me know if this is an acceptable format or if I should change it. There are multiple pages.

1 Basic Operations

1.1 Some Calculations

Let $a = 2 + 3i$ and $b = 3 - 4i$. Simplify the following expressions, by expressing them in the form $z = x + iy$, for real numbers x, y.

- $z = a + 3b$
  $= (2 + 3i) + 3(3 - 4i)$
  $= 2 + 3i + 9 - 12i$
  $= \mathbf{11 - 9i}$

- $z = 1 + ab$
  $= 1 + (2 + 3i)(3 - 4i)$
  $= 1 + (6 - 8i + 9i + 12)$
  $= \mathbf{19 + i}$

- $z = \overline{a + b}$
  $= \overline{(2 + 3i) + (3 - 4i)}$
  $= \overline{5 - i}$
  $= \mathbf{5 + i}$

- $z = |a + b|$
  $= |(2 + 3i) + (3 - 4i)|$
  $= |5 - i|$
  $= \sqrt{5^2 + (-1)^2}$
  $= \sqrt{26}$
  $\approx \mathbf{5.100}$

- (**Extra Credit**) $z = \frac{a}{b} - \frac{b}{a}$
  $= \frac{2+3i}{3-4i} - \frac{3-4i}{2+3i}$
  $= \left(\frac{2+3i}{3-4i}\right)\left(\frac{3+4i}{3+4i}\right) - \left(\frac{3-4i}{2+3i}\right)\left(\frac{2-3i}{2-3i}\right)$
  $= \frac{(2+3i)(3+4i)}{(3)^2-(4i)^2} - \frac{(3-4i)(2-3i)}{(2)^2-(3i)^2}$
  $= \frac{6+8i+9i-12}{9+16} - \frac{6-9i-8i-12}{4+9}$
  $= \frac{-6+17i}{25} - \frac{-6-17i}{13}$
  $= \frac{13(-6+17i)}{(13)(25)} - \frac{25(-6-17i)}{(13)(25)}$
  $= \frac{-78+221i}{325} - \frac{-150-425i}{325}$
  $= \mathbf{\frac{72+646i}{325}}$

1.2 Complex Norm

Prove that for an arbitrary complex number $z, |z|^2 = z\overline{z}$.

(Just going to rewrite some definitions here.)

$z = a + bi$ for two arbitrary real numbers a, b, and its conjugate $\overline{z} = a - bi$. $|z|$ is defined to be the distance from the origin to the point z in the complex plane - that is, $\sqrt{a^2 + b^2}$. Thus:

$|z|^2$
$= \left(\sqrt{a^2 + b^2}\right)^2$
$= a^2 + b^2$

Similarly, $\overline{z} = a - bi$, so:

$z\overline{z}$
$= (a + bi)(a - bi)$
$= a^2 - abi + abi - b^2i^2$
$= a^2 + b^2$

Since $|z|^2$ and $z\overline{z}$ are both equal to $a^2 + b^2$, $\mathbf{|z|^2 = z\overline{z}}$.

2 Euler's Formula

Euler's formula is $e^{i\theta} = \cos\theta + i\sin\theta$.

2.0.1 Part (a)

Evaluate the following exponents:

- $e^{i\pi}$
  $= \cos\pi + i\sin\pi$
  $= 1 + 0i$
  $= \mathbf{1}$

- $e^{i\frac{\pi}{2}}$
  $= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
  $= 0 + 1i$
  $= \mathbf{i}$

- $e^{-i\frac{\pi}{6}}$
  $= \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$
  $= \boldsymbol{\frac{\sqrt{3}}{2}} - \boldsymbol{\frac{1}{2}}\mathbf{i}$

2.0.2 Part (b)

Show that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{1}$$

What would be the analogous expression for $\sin x$?

Expanding the right side of the equation:

$$\frac{e^{ix} + e^{-ix}}{2} = \frac{(\cos x + i\sin x) + (\cos(-x) + i\sin(-x))}{2}$$

Conveniently, we know $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, so:

$$\frac{(\cos x + i\sin x) + (\cos(-x) + i\sin(-x))}{2} = \frac{\cos x + i\sin x + \cos(x) - i\sin(x)}{2}$$

We can see that the imaginary parts cancel out, leaving us with:

$$\frac{\cos x + i\sin x + \cos(x) - i\sin(x)}{2} = \frac{2\cos x}{2} = \cos x$$

as intended!

To find the analogous expression for $\sin x$, we aim to cancel the real part while retaining the imaginary part, as follows:

$$\sin x = \frac{2i\sin x}{2i} \tag{2}$$

Multiplying in the i accounts for the fact that $\sin x$ has a coefficient of i in Euler's formula. Proceeding by adding cancelling $\cos x$ terms:

$$\frac{2i\sin x}{2i} = \frac{\cos x + i\sin x - \cos x + i\sin x}{2i}$$

Grouping some terms gives us:

$$\frac{\cos x + i\sin x - \cos x + i\sin x}{2i} = \frac{(\cos x + i\sin x) - (\cos x - i\sin x)}{2i}$$

Again invoking the fact that $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$:

$$\frac{(\cos x + i\sin x) - (\cos x - i\sin x)}{2i} = \frac{(\cos x + i\sin x) - (\cos(-x) + i\sin(-x))}{2i}$$

Now we can invoke Euler's formula:

$$\frac{(\cos x + i\sin x) - (\cos(-x) + i\sin(-x))}{2i} = \frac{e^{ix} - e^{-ix}}{2i}$$

Therefore, we find our expression for $\sin x$:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{3}$$

2.0.3 Part (c)

Write $z = 1 - \sqrt{3}i$ in polar form, then evaluate z^6.

Using the formula $z = |z|e^{i\theta}$, we can find $|z|$ and θ:

$$|z| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

Therefore, $\mathbf{z = 2e^{-i\frac{\pi}{3}}}$. Now we can find z^6 much more easily:

$$z^6 = \left(2e^{-i\frac{\pi}{3}}\right)^6 = 2^6 e^{-i\frac{6\pi}{3}} = 64e^{-2i\pi} = 64(\cos(-2i\pi) + i\sin(-2i\pi)) = \mathbf{64}$$

2.0.4 Part (d)

Try to prove the sine and cosine addition formula using Euler's formula. You need to show

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

Immediately from using Euler's formula on the angle $x+y$, we see that:

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y) \tag{4}$$

So we look for an equivalent expression where the real and imaginary parts are equivalent to the right side of the addition formulae. We can use Euler's formula on x and y to find:

$$e^{ix} = \cos x + i\sin x$$
$$e^{iy} = \cos y + i\sin y$$

We can multiply these two expressions to get an expression for $e^{i(x+y)}$:

$$e^{ix}e^{iy} = e^{i(x+y)} = (\cos x + i\sin x)(\cos y + i\sin y)$$

Inconveniently long, this expands to:

$$\cos x \cos y + i\cos x \sin y + i\sin x \cos y - \sin x \sin y$$

Rearranging the terms and grouping gives us:

$$(\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y) \tag{5}$$

Equating the real and imaginary parts of (4) and (5) gives us the addition formulae, as intended:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$