Linear Algebra Assignment: Homework

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1 Matrix Operations

Reference: $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, b = \begin{pmatrix} 1 \\ i \end{pmatrix}, H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.

Part 1 concerns review of matrix multiplications and the definition of unitary matrices, used to represent quantum states.

- $Ha = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
  $= \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$
  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 + (-1) \cdot 0 \end{pmatrix}$
  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
  $= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

- $Yb = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$
  $= \begin{pmatrix} 0 \cdot 1 + i \cdot i \\ -i \cdot 1 + 0 \cdot i \end{pmatrix}$
  $= \begin{pmatrix} -1 \\ -i \end{pmatrix}$

- $H^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^T$
  $= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

- $Y^T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^T$
  $= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

- $\overline{H} = \overline{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}$
  $= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

- $\overline{Y} = \overline{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}$
  $= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

- For the next two, I use the definitions of $\overline{H}$ and $\overline{Y}$ from above.

- $H^\dagger = \overline{H}^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^T$
  $= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

- $Y^\dagger = \overline{Y}^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^T$
  $= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

- For the next two, I use the definitions of $H^\dagger$ and $Y^\dagger$ from above.

- $H^\dagger H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
  $= \begin{pmatrix} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \end{pmatrix}$
  $= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{pmatrix}$
  $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- $Y^\dagger Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
  $= \begin{pmatrix} 0 \cdot 0 + i \cdot (-i) & 0 \cdot i + i \cdot 0 \\ -i \cdot 0 + 0 \cdot (-i) & -i \cdot i + 0 \cdot 0 \end{pmatrix}$
  $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- $a \cdot b = \overline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$
  $= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$
  $= 1 \cdot 1 + 0 \cdot i$
  $= 1$

- (**Extra Credit**) Show that unitary matrices preserve inner products.

  Let U be a unitary matrix such that $U^\dagger U = \mathbb{I}$, the identity matrix. Then, for two arbitrary vectors $\vec{v_1}, \vec{v_2}$ let $\vec{w_1} = U\vec{v_1}$ and $\vec{w_2} = U\vec{v_2}$. Then:
  $\vec{w_1} \cdot \vec{w_2} = (U\vec{v_1}) \cdot (U\vec{v_2})$
  $= (U\vec{v_1})^\dagger (U\vec{v_2})$
  $= (U^\dagger \vec{v_1}^\dagger)(U\vec{v_2})$
  $= (U^\dagger U)(\vec{v_1}^\dagger \vec{v_2})$
  $= \mathbb{I}(\vec{v_1} \cdot \vec{v_2})$
  $= \boldsymbol{\vec{v_1} \cdot \vec{v_2}}$

2 Linear Basis (Extra Credit)

2.1 Part (a)

Give examples of:

- A linear space: $\mathbb{R}^2$
- A set of linearly dependent vectors living in this space: $\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
- A set of linearly independent vectors living in this space: $\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \end{pmatrix}$
- A basis for this vector space: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

2.2 Part (b)

Which of the following sets in $\mathbb{R}^2$ form a basis?

- $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$: Yes, since they are orthogonal, as $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 1 \cdot 0 + 0 \cdot 2 = 0$, and therefore must be linearly independent. We can also see that no scalar quantity can multiply the 0 in the second vector to make it equal to the 1 in the first, or vice versa for the 2 in the second vector.

- $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$: No, it is easy to see that $\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$: No, as we immediately see there are three vectors in this set - the most that can form a basis is $\dim(\mathbb{R}^2) = 2$. It's also obvious that the third is the sum of the first two.

- $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$: Yes, surprisingly! While they are not orthogonal, they are linearly independent, as we can see that no scalar quantity exists that can multiply the 0 in the first vector to make it equal to the 1 in the second - neither vector can be written as a linear combination of the other.

2.3 Part (c)

Write the elements of the canonical basis, $\vec{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, as linear combinations of $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Find the projections of $\vec{e_1}$ and $\vec{e_2}$

onto $\vec{u}$ and $\vec{v}$.

- $\vec{e_1} = \frac{1}{2}(\vec{u} + \vec{v})$
- $\vec{e_2} = \frac{1}{2}(\vec{u} - \vec{v})$
- We use outer products to determine projections, but we need to normalize $\vec{u}$ and $\vec{v}$ first: their norms are both $\sqrt{2}$, so $\vec{u_n} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\vec{v_n} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$
- $|u\rangle\langle u| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$
  $|v\rangle\langle v| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$
- $\vec{e_1}$ onto $\vec{u}$: $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$
- $\vec{e_1}$ onto $\vec{v}$: $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$
- $\vec{e_2}$ onto $\vec{u}$: $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$
- $\vec{e_2}$ onto $\vec{v}$: $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

2.4 Part (d)

For what value(s) of x will the following set of vectors **not** form a basis for $\mathbb{R}^3$? For those values, which elements of $\mathbb{R}^3$ cannot be written as a linear combination of the vectors?

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ x \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix} \right\} \tag{1}$$

We clearly see $x = 0$ makes this set not form a basis - there are no components in the third dimension that can make anything, so all vectors with a third (z) component equal to 0 in $\mathbb{R}^3$ *cannot* be written as a linear combination when $x = 0$. We can also see that $x = -1$ makes the set not form a basis, as the first vector is the sum of the last two, and therefore is linearly dependent on them.

However, can we prove this more rigorously? Using the knowledge (which admittedly wasn't in the text) that we can check for linear independence by taking the determinant of the matrix formed by the vectors, we can gain exact values for x that make this set not form a basis. Strictly, when the determinant is zero, the set is linearly dependent, since the product $\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & x \\ 1 & x & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has solutions for nonzero a, b, and/or c (coefficients of a system of linear equations that relate the vectors). The determinant $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & x \\ x & x & 0 \end{vmatrix} = 1(1 \cdot 0 - x \cdot x) - 0(0 \cdot 0 - x \cdot x) + 1(0 \cdot x - 1 \cdot x) = -x^2 - x = -x(x+1)$, and setting this equal to 0 yields $x = 0, -1$ as expected. No other values of x will produce a determinant of 0, and therefore no other values of x will make the set not form a basis; our initial intuition covered all cases!

In addition, while we mentioned the vectors that cannot be written as a linear combination of this set for $x = 0$ (that is, all vectors with a nonzero third component), we never mentioned the vectors that cannot be written as a linear combination of this set for $x = -1$. Since the first vector is the sum of the latter two, we can remove it and only check the remaining two: $\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$. Thus, all vectors that can be written as a linear combination of these two vectors can be represented as $a \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ -a \end{pmatrix} + \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ a-b \\ -a \end{pmatrix}$, so any vector that cannot be represented as $\begin{pmatrix} b \\ a-b \\ -a \end{pmatrix}$ for $a, b \in \mathbb{R}$ cannot be written as a linear combination of this set of vectors for $x = -1$. For instance, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ fails.

2.5 Part (e)

Is a union of linear subspaces necessarily a linear subspace itself?

By my understanding of sets and unions, not necessarily. For instance, take the linear subspaces E_0 and E_1, the spans of the members of the canonical basis in $\mathbb{R}^2$, such that $E_0 = \left\{ \vec{v} = \begin{pmatrix} k \\ 0 \end{pmatrix} \mid k \in \mathbb{R} \right\}$ and $E_1 = \left\{ \vec{v} = \begin{pmatrix} 0 \\ k \end{pmatrix} \mid k \in \mathbb{R} \right\}$.

The union of these two linear subspaces $W = E_0 \cup E_1$ is not itself a linear subspace, as it contains only the vectors along each axis, but not the sums of any two vectors on different axes: for instance, W contains both $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, but not $\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$. Since W is not closed under addition, it is not a linear subspace itself.

2.6 Part (f)

Let V be a subspace of $\mathbb{R}^n$. Suppose that $S = \{v_1, v_2, \ldots, v_n\}$ is a spanning set of V. Prove that any set of $n+1$ or more vectors in V is linearly dependent.

Let our set of interest have cardinality r such that $r > n$. Then for every k such that $1 \leq k \leq r,\ k \in \mathbb{Z}$, we decompose $\vec{v_k}$ into components as follows:

$$\vec{v_k} = \begin{bmatrix} v_{1k} \\ v_{2k} \\ \vdots \\ v_{nk} \end{bmatrix} \tag{2}$$

Using the same knowledge of determinants as in Part (d), we create a set of scalar coefficients (not to be confused with the coefficient matrix) $c_1, c_2, \ldots, c_r \in \mathbb{R}$ such that $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_r\vec{v_r} = \vec{0}$ and set up the matrix equation (representing a system of linear equations):

$$\begin{bmatrix} v_{11} & v_{12} & \ldots & v_{1r} \\ v_{21} & v_{22} & \ldots & v_{2r} \\ \vdots & \vdots & & \vdots \\ v_{n1} & v_{n2} & \ldots & v_{nr} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3}$$

Like in part (d), if *any* member of the second matrix is nonzero, the set is linearly dependent. However, note that at least one solution must exist - the trivial solution, $c_1 = c_2 = \cdots = c_r = 0$.

Considering that this matrix multiplication represents a set of linear equations (for example, multiplying the first row gives $v_{11}c_1 + v_{12}c_2 + \cdots + v_{1r}c_r = 0$), we notice that there are more unknowns (r) than there are equations (n), since $r > n$. Thus, there are either no solutions or infinite solutions.

However, we already know at least one solution must exist; therefore, there are free variables in the system where manipulating them makes no change to the solution, and there are infinite solutions. Since these manipulations correspond to changing a term in the second matrix, this means there can be nonzero members of the second matrix, thus the set is linearly dependent.