Postulates of Quantum Mechanics Assignment: Homework

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1 Measurement Example

The ”computational basis” of qubit states is given by $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Another basis is given by $|\alpha\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $|\beta\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

1.1 Part (a)

Check that both bases are orthonormal.

- $|0\rangle \cdot |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$
- $|\alpha\rangle \cdot |\beta\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ i \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(1 - i^2) = 0$
- $||\vec{0}|| = \sqrt{|0|^2 + |1|^2} = \sqrt{1} = 1$
- $||\vec{1}|| = \sqrt{|1|^2 + |0|^2} = \sqrt{1} = 1$
- $||\vec{\alpha}|| = \sqrt{\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{i}{\sqrt{2}}\right|^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$
- $||\vec{\beta}|| = \sqrt{\left|\frac{1}{\sqrt{2}}\right|^2 + \left|-\frac{i}{\sqrt{2}}\right|^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$

1.2 Part (b)

Say our qubit was initially in state $|0\rangle$ and was measured in the basis of $|\alpha\rangle$ and $\beta\rangle$. What will the state of the qubit be immediately following the measurement? What are the probabilities of the outcomes?

We apply Born's rule: the state can collapse to either $|\alpha\rangle$ with probability $|\langle\alpha|0\rangle|^2$, or $|\beta\rangle$ with probability $|\langle\beta|0\rangle|^2$. Calculating these two gives $|\langle\alpha|0\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$, and $|\langle\beta|0\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$. Therefore, the state of the qubit will be $|\alpha\rangle$ with probability $\frac{1}{2}$ or $|\beta\rangle$ with probability $\frac{1}{2}$.

1.3 Part (c)

After the first measurement, the qubit is measured again in the basis $|0\rangle$, $|1\rangle$. What are the probabilities of the two outcomes now?

We consider all four possibilities: the qubit is measured in $|\alpha\rangle$ then $|0\rangle$, $|\alpha\rangle$ then $|1\rangle$, $|\beta\rangle$ then $|0\rangle$, and $|\beta\rangle$ then $|1\rangle$. We calculate the probability of each of these four possibilities, multiply each by the corresponding one of the first two, and sum them up.

- $|\langle 0|\alpha\rangle|^2 = \left|\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$
- $|\langle 1|\alpha\rangle|^2 = \left|\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}\right|^2 = \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$
- $|\langle 0|\beta\rangle|^2 = \left|\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$
- $|\langle 1|\beta\rangle|^2 = \left|\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}\right|^2 = \left|-\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$

We see that the probability of each outcome is equal; multiplying each by the $\frac{1}{2}$ original probability (of it collapsing into either $|\alpha\rangle$ or $|\beta\rangle$) and summing them up gives $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} = 1$ for total probability, as expected. The probability of it ending in state $|0\rangle$ is $\frac{1}{4}+\frac{1}{4} = \frac{1}{2}$, and state $|1\rangle$ also has probability $\frac{1}{4}+\frac{1}{4} = \frac{1}{2}$.

2 Distinguishing Between States

Two boxes each produce a stream of qubits. Box A produces qubits in the state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$. Box B randomly produces qubits in states $|0\rangle$ and $|1\rangle$, each with probability $\frac{1}{2}$. We have one of the boxes, but do not know which it is. Describe an experiment on the qubits that can tell the difference between box A and box B. Can you tell the difference between the boxes by examining only one of the qubits?

It's immediately obvious that measuring the qubits in the $|0\rangle$, $|1\rangle$ basis will not distinguish between the two boxes - Box B produces each with probability $\frac{1}{2}$, and some quick calculations show Box A produces each with probability $\frac{1}{2}$ as well. This is due to the inherent randomness in each; however, they are different types of random! Box A is *quantum* randomness, a superimposed state, whereas Box B is *classical* randomness, like a coin flip.

If we measure in a different basis system, perhaps we'll see something different. We notice that $|\Psi\rangle = |\alpha\rangle$, so what if we measure in the $|\alpha\rangle, |\beta\rangle$ basis? The qubits from Box A will always collapse to $|\alpha\rangle$, whereas the qubits from Box B will collapse to $|\alpha\rangle$ with probability $\frac{1}{2}\left(|\langle\alpha|0\rangle|^2 + |\langle\alpha|1\rangle|^2\right) = \frac{1}{2}(\left|\frac{1}{\sqrt{2}}\right|^2 + \left|-\frac{i}{\sqrt{2}}\right|^2) = \frac{1}{2}(1) = \frac{1}{2}$ and $|\beta\rangle$ with (by similar math) probability $\frac{1}{2}$. Therefore, if we measure in the $|\alpha\rangle, |\beta\rangle$ basis, we can distinguish between the two boxes - the box from which there are qubits measured in state $|\beta\rangle$ is Box B.

However, since Box B also has a $\frac{1}{2}$ chance of emitting a qubit measured in state $|\alpha\rangle$, we cannot always determine which box is which by examining only one qubit. If the one qubit is measured in state $|\beta\rangle$, we know the box it originated from is Box B, but if it is measured in state $|\alpha\rangle$, we cannot tell which box it came from.

3 (Extra Credit) Mach-Zehnder Interferometer

- Say the phase shifter is absent and a photon enters the interferometer from below, described by a state vector $|v\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The action of the left beamsplitter can be represented by the matrix $B_l = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and the right by $B_u = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. The final state will be represented by $B_u(B_l|v\rangle)$ - the action of B_l on $|v\rangle$, then B_u applied to that. Thanks to matrix associativity, we can instead first evaluate B_uB_l and apply that transformation to $|v\rangle$ - specifically, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} =$

$\frac{1}{2}\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. Applying this yields $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \cdot \begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$, so the final state is $|v'\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}$. This corresponds to a $|1|^2 = 1$ probability of detection by D_0 and $|0|^2 = 0$ probability of detection by D_1.

- Now consider the same problem but with a phase shifter present in the lower beam, such that the phase of the lower beam component is shifted by π. Its effect on the state of the photon is given by the matrix $P = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. With our photon again starting in state $|v\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$, the final state is now given by $B_u(P(B_l|v\rangle)) = (B_u(PB_l))|v\rangle$, again thanks to matrix associativity. Note that we cannot move around the matrix order, since matrix multiplication is not commutative! We calculate $P \cdot B_l$ as $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \cdot \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$. Multiplying B_u by this gives $\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-2 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. Applying this yields $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \cdot \begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$, so the final state is $|v'\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$. This corresponds to a $|0|^2 = 0$ probability of detection by D_0 and $|1|^2 = 1$ probability of detection by D_1.

- Now say we have a phase shift by an arbitrary phase ϕ, with its effect on the state of the photon given by the matrix $P = \begin{pmatrix}1 & 0\\ 0 & e^{i\phi}\end{pmatrix}$. Again, the final state is $B_u(P(B_l|v\rangle)) = (B_u(PB_l))|v\rangle$, and we first calculate $P \cdot B_l = \begin{pmatrix}1 & 0\\ 0 & e^{i\phi}\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & e^{i\phi}\end{pmatrix} \cdot \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ e^{i\phi} & -e^{i\phi}\end{pmatrix}$. Then, multiplying B_u by this yields $\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ e^{i\phi} & -e^{i\phi}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 1\\ e^{i\phi} & -e^{i\phi}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}e^{i\phi}-1 & -e^{i\phi}-1\\ e^{i\phi}+1 & -e^{i\phi}+1\end{pmatrix}$. Operating on $|v\rangle$ with this, we get: $\frac{1}{2}\begin{pmatrix}e^{i\phi}-1 & -e^{i\phi}-1\\ e^{i\phi}+1 & -e^{i\phi}+1\end{pmatrix} \cdot \begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-e^{i\phi}-1\\ -e^{i\phi}+1\end{pmatrix}$. The probability of D_0 measuring the photon is $\left|\frac{-e^{i\phi}-1}{2}\right|^2$, and expanding this using Euler's formula gives $\left|-\frac{\cos\phi+1}{2} - \frac{\sin\phi}{2}i\right|^2$. Calculating the square of the absolute value as a^2+b^2, this comes out to $\left(\frac{\cos\phi+1}{2}\right)^2 + \left(\frac{\sin\phi}{2}\right)^2 = \frac{\cos^2\phi+2\cos\phi+1+\sin^2\phi}{4} = \frac{2\cos\phi+2}{4} = \frac{\cos\phi+1}{2}$ (minus signs removed for ease of calculation throughout).

Doing similar calculations for D_1 yields $\left|\frac{-e^{i\phi}+1}{2}\right|^2 = \left|-\frac{\cos\phi - 1}{2} - \frac{-\sin\phi}{2}\right|^2 = \left(\frac{\cos\phi-1}{2}\right)^2 + \left(\frac{\sin\phi}{2}\right)^2 = \frac{\cos^2\phi - 2\cos\phi + 1 + \sin^2\phi}{4} = \frac{-2\cos\phi+2}{4} = \frac{-\cos\phi+1}{2}$. To reiterate, the chance of D_0 measuring the photon is $\frac{\cos\phi+1}{2}$, and the chance of D_1 measuring the photon is $\frac{-\cos\phi+1}{2}$. These sum to 1, as expected: $\frac{\cos\phi+1}{2} + \frac{-\cos\phi+1}{2} = \frac{\cos\phi - \cos\phi + 2}{2} = \frac{2}{2} = 1$.

- Now say the phase shifter is gone but the photon enters the interferometer in a superposition $|v\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Without the phase shifter, our final state is given again by $(B_u B_l)|v\rangle$, as in the first question: we can reuse our value for $B_u B_l$, that being $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Applying this transformation to $|v\rangle$ yields $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ -\alpha \end{pmatrix}$. As such, the probability of D_0 measuring the photon is $|\beta|^2$, and the probability of D_1 measuring the photon is $|-\alpha|^2 = |\alpha|^2$.