STEM Notes

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Preface

About Me

Hi! I'm Christian Zhou-Zheng, high school class of 2026 at The Pingry School, located in Basking Ridge, New Jersey. I'm very STEM-focused, and I created this document in freshman year to keep track of all my high school notes for such classes. I'm taking the most accelerated track possible in each STEM field offered by the school (math, science, computer science), so this document should hopefully still prove useful to others who are in the upper grades/taking highly rigorous courses.

How to Use This Document

Firstly: This is **NOT** intended to teach you material. This is a study guide. It evolved from a collection of "cheat sheets" of formulas and ideas, and its goal is to provide something to brush up with when studying - just enough to stir your memory around that concept.

This is a collection of my notes from various STEM classes and personal research, first collated in the spring of freshman year and therefore not containing absolutely everything. I will attempt to update this on a rolling basis so all future topics will be covered, but I can't be bothered adding in all past topics. In addition, several sections are intentionally made to fit on only one page, for ease of printing. However, some places may include external links to other resources, which may be more in-depth.

Some important topics from just before this document was created (particularly basic algebra and geometry/trigonometry) will be added, but the rest won't. I'm also not currently particularly willing to make or add graphs for most of these topics, nor the derivations - just the formulas. In particular, the following will be **missing** from this document:

- Basic algebra (polynomials, functions, etc)
- **Almost all** geometry
- Basic trigonometry (SOHCAHTOA level stuff, simple definitions)
- **All** pre-calculus math up to limits
- **Most** early chemistry (pre-AP)
- **All** biology
- Basic computer science up to and including the AP Computer Science A curriculum

Part I

Math

1 Precalculus - Honors

WARNING: This does not cover material before limits.

1.1 Limits

Limits are basically asking what the value of a point on a graph is as you get infinitely close to that point from either side. We can think of a limit as the value we *expect* a function to be at at a certain x-value, by extrapolating from its behavior on either side, even if the function isn't necessarily equal to that value at that x-value. We denote this for a function $f(x)$ where the point in question is a as:

$$\lim_{x \to a} f(x)$$

An important question to ask first is that of **continuity**, where the definition for a graph being continuous at $x = a$ is:

$$f(a) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

where $\lim_{x \to a^- f(x)}$ is the limit of $f(x)$ approaching from the negative direction, and vice versa for the positive. Basically, what this means is that $f(x)$ doesn't jump around or move unpredictably around point a - it comes smoothly in, passes through point a, and goes smoothly out. This will be important when we get to *derivatives*, which are closely related to limits.

Similarly, the limit from *both* directions - that is, $\lim_{x \to a} f(x)$ - only exists when the following is true:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Notice how $f(x)$ doesn't need to be equal - that means there can be a hole/discontinuity of the function there and the limit will *still exist!* I would add graphs to visualize this but I'm too lazy.

It's important to note, however, that while a can go to infinity, the *limit itself* cannot; you cannot have the limit of a fuction be infinity (which makes sense when you think about it).

The first method you should always try to use to evaluate a limit is substitution - that is, just plugging in a for x in $f(x)$. This will allow you to identify and solve the two trivial types of limits.

The first trivial type: substituted away

The first trivial type can be solved with simple substitution for x - really, you don't even need to write this as a limit. For example, take $f(x) = x^2 + 2x - 3$; evaluating $\lim_{x \to 2} f(x)$ yields $\lim_{x \to 2}[(2)^2 + 2(2) - 3]$, which becomes just $\lim_{x \to 2}[5]$, and the limit of a constant is just that constant - it won't change with x! As such, this limit was no harder than a standard polynomial to solve.

The second trivial type: Does Not Exist/DNE

The second trivial type can easily be identified by substitution; a limit not existing can be seen easily when the graph of the limit goes to ∞ or $-\infty$. For example, $\lim_{x\to 0} 1/x$ yields $\frac{1}{0}$, clearly inappropriate! As we'll see soon, the limit is only DNE when the answer turns out to be a number over 0 (or infinity); $\frac{0}{0}$ is surprisingly existent, as are $\frac{\infty}{0}$ and so on.

Non-trivial Limits and Indeterminate Form

If, after substitution, you get the answer $\frac{0}{0}$ or $\frac{\infty}{0}$ or so on, congratulations! You have to utilize your neurons to work out a more accurate answer than this **indeterminate form** of the limit. Usually, this happens because a is a factor of both the numerator and denominator; try various methods, such as factoring, conjugation, combining fractions, etc. to reduce the function within the limit to a point where substitution will provide a reasonable answer.

For example, evaluating $lim_{x\to 3}\left[\frac{x^2-9}{x-3}\right]$ results in attempting to take $lim_{x\to 3}\left[\frac{0}{0}\right]$, which means there's still more to be done! In this case, we can factor the numerator and cancel a $x-3$ term from the numerator and denominator, resulting in the evaluatable $\lim_{x\to 3}(x+3) = 6$ via substitution.

Do note that it is **fallacious** to call the functions you get equal, although the limits are. Apparently the AP Calc test likes to take points off for this. For example, $\lim_{x\to 3}\left[\frac{x^2-9}{x-3}\right] = \lim_{x\to 3}[x+3]$, but $\frac{x^2-9}{x-3} \neq x+3$!

1.1.1 One-Sided Limits

Recall our definitions of continuity and limit existence. In particular, the limit existence from *both* directions:

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

However, even if this doesn't hold true, we can still evaluate the limit from *one* direction. This is useful for points where:

- the function makes a "jump"
- the function is not defined in one direction
- the function makes a sharp turn

Examples include $\lim_{x\to 0}\sqrt{x}$, which must be evaluated from the positive direction, and $\lim_{x\to 0}\frac{|x|}{x}$, which can be evaluated from either direction. In the event that the limit must be evaluated only from one direction, we must determine *which* direction to evaluate it from. Usually, this takes the form of a sign chart.

1.1.2 Limit Rules

$$\lim_{x \to a}[c * f(x)] = c * \lim_{x \to a}[f(x)]$$

$$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a}[f(x)] \pm \lim_{x \to a}[g(x)])$$

$$\lim_{x \to a}[f(x) * g(x)] = \lim_{x \to a}[f(x)] * \lim_{x \to a}[g(x)]$$

$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a}[f(x)]}{\lim_{x \to a}[g(x)]}$$

$$\lim_{x \to a}[f(x)]^n = [\lim_{x \to a} f(x)]^n$$

1.1.3 Limits to Infinity

For a rational function $f(x)$, $lim_{x \to \infty} f(x)$ depends on the *highest degree term* in the numerator and denominator of the function $f(x)$.

- If the degree of the numerator is higher than the degree of the denominator, the limit evaluates to ∞ (or $-\infty$ where applicable).
- If the degree of the denominator is higher than the degree of the numerator, the limit evaluates to 0.
- If the degrees are equal, the limit evaluates to the *quotient of the leading term coefficients.*

1.1.4 Sums of Infinite Geometric Series

Yeah, I said I wouldn't put AP/GPs in here but this counts as limits. tl;dr:

$$\sum_{k=1}^{\infty} a_1 r^{k-1} = \lim_{n \to \infty} \sum_{k=1}^{n} a_1 r^{k-1} = \lim_{n \to \infty}\left[\frac{a_1 - a_1 r^n}{1 - r}\right]$$

for a_1 being the starting term and r the common ratio.

1.2 Precalculus Trigonometry

This is just a list of the formulas you learn in precalc trig. Polar coordinates are skipped because I can't be bothered with graphs.

1.2.1 Unit Circle Values

All values are in radians.

	$\theta = 0$	$\theta = \frac{\pi}{6}$	$\theta = \frac{\pi}{4}$	$\theta = \frac{\pi}{3}$	$\theta = \frac{\pi}{2}$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N/A

1.2.2 Trigonometric Identities

Basically everything can be derived from the Pythagorean identity and a few others.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

1.2.3 Trigonometric Formulas

Angle Addition Formulas:

$$\sin(\alpha \pm \beta) = \sin\alpha\,\cos\beta \pm \cos\alpha\,\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\,\cos\beta \mp \sin\alpha\,\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\,\tan\beta}$$

Double Angle Formulas: Derived by plugging in θ for α and β in the angle addition formulas.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\tan(2\theta) = \frac{2\,\tan\theta}{1 - \tan^2\theta}$$

Half Angle Formulas: Derived by plugging in $\frac{1}{2}\theta$ to the double angle formulas and solving.

$$\sin\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

1.3 Non-Right-Triangle Trigonometry

Let triangle $\triangle ABC$ have sides a, b, c where the side with a lowercase letter is opposite the angle marked with the corresponding uppercase letter. This triangle must not necessarily be right.

These formulas are derived from splitting a triangle into two right triangles by using the height from an adjacent angle, then using trigonometric functions/identities on that.

1.3.1 Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Ambiguous Case: Since $\sin A$ is positive over all $x \in [0, 180]$ degrees, there can be times where $\sin A$ *can take on two possible values when solving using the Law of Sines.* In particular, this happens in a triangle with two sides and an angle given in **Side-Side-Angle** format, in which angle A is not opposite the longest side and the angle measure given is less than both A and $180 - A$ after solving using Law of Sines.

1.3.2 Law of Cosines

Note that this is equal to the Pythagorean Theorem for $C = 90°$, at which point $\cos C = 0$.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

1.3.3 Area of a Triangle

Side-Angle-Side Formula:

$$A_\triangle = \frac{1}{2}ab\sin C$$

where a, b are adjacent and C is the angle between the two.

Side-Side-Side (Heron's) Formula:

$$A_\triangle = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{a+b+c}{2}$$

where s is also known as the **semiperimeter**.

1.4 Vectors

1.4.1 Precalculus Vectors

Honors Precalculus vectors are slightly different than physics vectors. I've written much on the latter, so there's significant overlap.

Vectors are defined by a **magnitude** (represented as $\|w\|$, also called the vector's *norm*) and **direction**. A vector goes from the **starting point** to the **terminal point**. They can be represented in geometric form ($\vec{A}$) or with bold lowercase letters ($\mathbf{u}$). Two vectors with the same magnitude and direction are equivalent, regardless of origin - *vectors contain no information about location!*

1.4.2 Component Form

Vectors are often written as if the starting point was the origin, represented only by the coordinates of its (adjusted) terminal point. This **component form** is denoted $\langle x_1, x_2 \rangle$. For a vector going from starting point (x_1, y_1) to terminal point (x_2, y_2), its component form is:

$$\langle x_2 - x_1, y_2 - y_1 \rangle$$

1.4.3 Linear Combinations

For vector addition, please see the physics section on vectors. Multiplying a vector and scalar multiplies all of the individual components of the vector by the scalar (1D) quantity. Combining these produces **linear combinations** of vectors. For example, for two vectors $\mathbf{u} = \langle 4, 1 \rangle, \mathbf{v} = \langle 2, 3 \rangle$:

$$\mathbf{u} + \mathbf{v} = \langle (4+2), (1+3) \rangle = \langle 6, 4 \rangle$$
$$2\mathbf{u} = \langle (2*4), (2*1) \rangle = \langle 8, 2 \rangle$$
$$2\mathbf{u} + 3\mathbf{v} = \langle 8, 2 \rangle + \langle 6, 9 \rangle = \langle 14, 11 \rangle$$

Vectors can also have more than two components; they can have as many as you want! However, you can only operate between vectors of the same dimension (same number of components), except for 1D scalar quantities, which can be multiplied against any dimension of vector. Operations on these linear combinations through vector addition and scalar multiplication follow the same rules as arithmetic addition and multiplication.

1.4.4 Unit Vectors

A **unit vector** is a vector of magnitude 1 in the direction of another vector $\vec{v}$, denoted $\hat{v}$ ("v-hat"). The unit vector of any vector can be found by dividing the vector by its magnitude; for example, the vector $\langle 3, 4 \rangle$ has a magnitude of 5 and unit vector $\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

The unit vectors parallel to the axes are called the **standard unit vectors** and denoted $\hat{\imath}$ and $\hat{\jmath}$ in the x- and y-directions, respectively. Any vector in component form can be written as a linear combination of the standard unit vectors. Similarly, vectors can be expressed using their magnitudes and angles. Examples of both follow:

$$\langle x, y \rangle = x\hat{\imath} + y\hat{\jmath}$$
$$\vec{v} = \langle \|v\| \cos\theta, \|v\| \sin\theta \rangle = \|v\|(\cos\theta)\hat{\imath} + \|v\|(\sin\theta)\hat{\jmath}$$

1.4.5 Vector Multiplication: Dot Product

One form of vector multiplication is called the **dot product**, which results in a *scalar* quantity. The dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is denoted $\mathbf{u} \cdot \mathbf{v}$ and is defined as:

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1 v_1 + u_2 v_2$$

Deriving from the Law of Cosines, treating the sides of a triangle as vectors, we also get:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

where θ is the angle between the vectors. Therefore, solving for θ results in:

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\sqrt{\mathbf{u} \cdot \mathbf{u}} \sqrt{\mathbf{v} \cdot \mathbf{v}}} \right)$$

The dot product is commutative and distributive. The dot product of a vector and itself is also equivalent to the *magnitude* of the vector, regardless of how many dimensions the vector has!

Orthogonal vectors are vectors that are perpendicular to each other. The dot product of two orthogonal vectors is 0, and the dot product of a vector and its unit vector is 1, since $\cos \theta$ is 0 when two vectors are perpendicular ($\theta = 90°$) and $\cos \theta$ is 1 when two vectors are parallel ($\theta = 0°$).

1.4.6 Vector Multiplication: Cross Product

This isn't in precalculus (you don't actually get here for years) but it's worth mentioning while we're at vectors. The other form of vector multiplication is called the **cross product**, which results in a vector that is perpendicular to *both* of the original vectors. The cross product is *only applicable in three dimensions.* The cross product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is denoted $\mathbf{u} \times \mathbf{v}$ and is defined as:

$$\mathbf{u} \times \mathbf{v} = \langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

Alternatively, you can use the magnitudes and angles of each vector, and the unit vector perpendicular to the plane containing both:

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin \theta \hat{n}$$

The perpendicular unit vector $\hat{n}$ can be found using the right hand rule: point your index finger in the direction of vector $\mathbf{u}$, your middle finger in the direction of vector $\mathbf{v}$, and your thumb will point in the direction of $\hat{n}$.

The cross product is anticommutative (that is, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$) and distributive. The cross product of a vector and itself is 0, and the cross product of a vector and its unit vector is the vector itself.

2 Calculus

2.1 Derivatives

Remember that the derivative of a function at a point is its **instantaneous rate of change** - that is,

$$f'(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

You could also describe it as the slope of the tangent line, the slope of a secant line as the distance between the intersection points approaches 0, etc. Usual notation for the derivative of $f(x)$ is $f'(x)$ (or $f(x)'$), pronounced "f prime of x," or $\frac{d}{dx}f(x)$, meaning "the derivative of f with respect to x."

2.1.1 Derivative Identities

For some constant a:

$$\frac{d}{dx}a = 0$$

$$\frac{d}{dx}x = 1$$

$$\frac{d}{dx}ax = a$$

$$\frac{d}{dx}(af(x)) = a\frac{d}{dx}f(x)$$

Exponential and Logarithmic Functions:

$$\frac{d}{dx}e^n = e^n$$

$$\frac{d}{dx}a^x = ln(a)a^x$$

$$\frac{d}{dx}ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}log_a(x) = \frac{1}{xln(a)}$$

Trigonometric Functions:

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\,\cot(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\,\tan(x)$$

$$\frac{d}{dx}\cot(x) = \csc^2(x)$$

Inverse Trigonometric Functions:

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$

2.1.2 Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

If you have the time, try deriving this yourself. (Hint: binomial expansion!)

2.1.3 Properties of Derivatives

For two functions f and g:

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$
$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

2.1.4 Chain Rule

The chain rule applies to compositions of functions, such that the derivative of $f(g(x))$ for two functions f and g is:

$$f(g(x))' = f'(g(x))g'(x)$$

Or in Leibniz's notation:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

2.1.5 Implicit Differentiation

Implicit differentiation is the differentiation of an equation in which y cannot be isolated - for example, $x^3+xy+y^3 = 1$. To differentiate implicitly, treat y as a function of x (and x as a function of x, for the purpose of things like the product rule) and differentiate both sides of the equation with respect to x, using the chain rule where necessary to introduce $\frac{dy}{dx}$ terms. The above equation would become $3x^2 + y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$. Then, solve for $\frac{dy}{dx}$.

2.2 (Indefinite) Integrals

Remember that there are two types of integrals - **definite** integrals, which can be thought of over the real plane as the area under a curve between two points a and b, and **indefinite** integrals, which are just antiderivatives. The **fundamental theorem of calculus** states that, for some function f with antiderivative F:

$$\int_a^b f(x) = F(b) - F(a)$$

So then it becomes a matter of finding said antiderivative. Notice that *nowhere* in this does anything on F between a and b matter!

2.2.1 Integral Identities

Basic Identities:

$$\int a\,dx = ax + C$$

$$\int c * f(x)\,dx = c * \int f(x)\,dx$$

$$\int \frac{1}{x}\,dx = ln|x| + C$$

Exponential and Logarithmic Functions:

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{ln(a)} + C$$

$$\int ln(x)\,dx = xln(x) - x + C$$

Trigonometric Functions:

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \tan(x)\,dx = ln|\sec(x) + C|$$

$$\int \csc(x)\,dx = ln|\csc(x) - \cot(x)| + C = ln|\tan(\frac{x}{2})| + C$$

$$\int \sec(x)\,dx = ln|\tan(x) + \sec(x)| + C$$

$$\int \cot(x)\,dx = ln|\sin(x)| + C$$

Inverting the others...

$$\int \sec^2(x)\,dx = \tan(x) + C$$
$$\int \csc^2(x)\,dx = -\cot(x) + C$$
$$\int \sec(x)\,\tan(x)\,dx = \sec(x) + C$$
$$\int \csc(x)\,\cot(x)\,dx = -\csc(x) + C$$

2.2.2 Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

2.2.3 Properties of Integrals

For two functions f and g:

$$\int (f \pm g)\,dx = \int f\,dx \pm \int g\,dx$$

2.2.4 Integration by Substitution

This is just the reverse chain rule. For two functions $f(x)$ and $g(x)$, write the equation as:

$$\int f(g)g'\,dx$$

And replace g with some temporary variable u. The equation then becomes:

$$\int f(u)\,du$$

which can be integrated and then u replaced by g after integration.

2.2.5 Integration by Parts

For two functions u and v:

$$\int u\,v\,dx = u\int v\,dx - \int u'\left(\int v\,dx\right)\,dx$$

2.2.6 Taylor Series

A **Taylor series** is a way to approximate a function $f(x)$ around a point a by using the function's derivatives. The Taylor series of $f(x)$ around a can be expressed as either of:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

3 Multivariable Calculus

3.1 Vector Calculus

This is copies of just formulas taken from *Introduction to Vector Analysis*, fifth edition, Davis and Snider. I won't explain them yet, but might add that later.

3.1.1 Vectors

$$\vec{A} \cdot \vec{B} = \|\vec{A}\| \|\vec{B}\| \cos\theta = \sum A_i B_i$$

$$\vec{A} \times \vec{B} = \|\vec{A}\| \|\vec{B}\| \sin\theta \hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$[\vec{A}, \vec{B}, \vec{C}] = \vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$B_\parallel = \frac{\vec{A} \cdot \vec{B}}{\vec{A} \cdot \vec{A}} \vec{A}$$

$$B_\perp = \frac{(\vec{A} \times \vec{B}) \times \vec{A}}{\vec{A} \cdot \vec{A}}$$

3.1.2 Scalar and Vector Fields

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\mathbf{grad}\ \phi = \nabla\phi$$

The gradient is the maximum rate of change and direction of that change of a scalar field ϕ at a point. It is a vector field.

$$\text{div}\ \vec{F} = \nabla \cdot \phi$$

The divergence is the net outflux of a vector field $\vec{F}$ per unit volume. It is a scalar field.

$$\mathbf{curl}\ \vec{F} = \nabla \times \vec{F}$$

The curl is the circulation of a vector field $\vec{F}$ per unit area. It is a vector field.

The integral theorems relating these:

$$\int_P^Q \nabla\phi \cdot d\vec{R} = \phi(Q) - \phi(P)$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \vec{n}\, dS$$

$$\iint_S \nabla \times \vec{F} \cdot \vec{n}\, dS = \int_C \vec{F} \cdot d\vec{R}$$

Vector identities and associated potential theorems:

$$\nabla \times \nabla\phi = 0 \ \text{ so } \ \nabla \times \vec{F} = 0 \implies \vec{F} = \nabla\phi$$

$$\nabla \cdot \nabla \times \vec{G} = 0 \ \text{ so } \ \nabla \cdot \vec{F} = 0 \implies \vec{F} = \nabla \times \vec{G}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \ \text{ so } \ \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$$

There's a lot more I could put in, but this is just from the reference sheet at the front of the book - considering I don't do multivariable calculus in high school, I won't be putting much of it here. I highly recommend the book - it's excellent if you have a background with vectors and partial derivatives.

Part II

Physics

4 AP Physics B

This is the material I studied freshman spring to skip Physics 1/Honors directly into AP Physics C - Mechanics. The textbook used was an old 2004 Physics B prep book, specifically Mooney's Physics B; I may reference diagrams from the text because I can't be bothered putting them in here.

4.1 1D Motion and Basic Kinematics

In basic kinematics, we can ignore air friction and round the force of gravity from $9.8m/s^2$ to $10m/s^2$. 1D motion involves three main concepts: **displacement**, **velocity**, and **acceleration**.

Displacement, Δx or just s, is a **measure of the change in x-position** between time t_0 and t (hence Δ for change). The standard AP unit is meters.

Velocity can be thought of as the derivative of displacement: the **rate of change of an object's position**. The standard AP unit is meters per second.

Acceleration can be thought of as the derivative of velocity, or the second derivative of displacement: the **rate of change of an object's velocity**, or the rate of change of the rate of change of an object's position. The standard AP unit is meters per second per second, or meters per second squared.

4.1.1 Symbols

The symbols typically used to represent a 1D system are:

- v - velocity
- v_0 - starting velocity at time t_0
- a - acceleration
- t - time
- Δx - displacement
- Δt - change in time from time t_0 to t
- Δv - change in velocity from time t_0 to t

4.1.2 Average/Instantaneous Velocity/Acceleration

The average velocity or acceleration over a period of time from t_0 to t is given as follows:

$$v_{avg} = \frac{\Delta x}{\Delta t}$$

$$a_{avg} = \frac{\Delta v}{\Delta t}$$

The instantaneous velocity or acceleration (derivative/tangent line!) is given as follows:

$$v_{inst} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$a_{inst} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

4.1.3 Constant Acceleration Equations

Acceleration is rarely zero; usually it is constant, and therefore the velocity changes over time. This slightly complicates things with measuring displacement, but luckily we have a system of four handy equations to help us with this:

$$\begin{cases} v = v_0 + at \\ \Delta x = \frac{1}{2}(v + v_0)t \\ \Delta x = v_0 t + \frac{1}{2}at^2 \\ v^2 = v_0^2 + 2a\Delta x \end{cases}$$

This is the order I learned them in and there's a good chance I'll refer back to them by number later on.

4.1.4 Law of Odd Numbers

This wasn't actually taught in the course but I stumbled across it doing practice problems.

Newton derived this law that states that **the ratio of distances traveled in equal times are proportional to the odd numbers**: e.g. a ball travels 5 m in the first second ($t_0 \to t_1$), 15 m in the second second ($t_1 \to t_2$), 25 m in the third second ($t_2 \to t_3$), and so on; these values are in the ratio 1 : 3 : 5 : .... This is handy for "an object traveled x distance in the tth second, how far will it travel in the nth second?" problems and the like.

4.2 2D Vectors

Physics vectors are slightly different than precalculus vectors. I've written much on the latter, so there's significant overlap.

A *vector* is a representation of 2D motion - it has a **magnitude**, which indicates its "length," and an **angle**, which indicates its direction (this angle is usually measured from the positive x-axis, like most trig). This is as opposed to *scalar* quantities, which only have a value - examples include velocity and acceleration.

Vectors are written as $\vec{A}$, while their magnitude is expressed as A (no arrow) and their angle as θ. The negative of vector $\vec{A}$ is a vector with the same angle θ but opposite magnitude.

4.2.1 Vector Components

As a 2D quantity, vectors naturally have x- and y-components. Although vectors are usually expressed in polar form, with a magnitude and angle (m, θ), they can also be simply separated into the x- and y- components, denoted as A_x and A_y:

$$\begin{cases} A_x = A\cos\theta \\ A_y = A\sin\theta \end{cases}$$

Of course, you can convert component form back into polar form, like so:

$$\begin{cases} A = \sqrt{A_x^2 + A_y^2} \\ \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) \end{cases}$$

with respect to the fact that $\tan^{-1}$ only produces values between $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the resulting angle may need to be corrected by a factor of π radians.

4.2.2 Vector Addition and Subtraction

Vector multiplication is not required to skip into AP Physics at Pingry, so that won't be covered here. However, vector addition is simple enough: you effectively lay the "tail" of one vector to the "head" of another, and chain this for as many vectors as you add; finally, determine the vector from the original tail to the final head. Notice that the order doesn't matter!

More simply, **we can just add the x-components and y-components individually**, adding two vectors $\vec{A}$ and $\vec{B}$ to create a vector $\vec{C}$:

$$\begin{cases} C_x = A_x + B_x \\ C_y = A_y + B_y \end{cases}$$

Similarly, subtraction can be done by subtracting $\vec{B}$'s x- and y-components from those of $\vec{A}$. Remember that subtracting is just adding the negative, and we know that $-\vec{B}$ is just $\vec{B}$ with the same angle but opposite magnitude.

$$\begin{cases} C_x = A_x - B_x \\ C_y = A_y - B_y \end{cases}$$

4.3 Projectile Motion

When evaluating projectile motion in the AP Physics curriculum, we simplify a lot. We disregard air resistance, so the x-velocity remains constant until the projectile hits a surface; the y-velocity is always the force of gravity, which on Earth is simplified to $10m/s^2$. As such, **we can calculate the horizontal and vertical components of the projectile's trajectory separately**. However, be careful to write with the correct subscripts for x- and y-displacement and velocity.

4.3.1 Horizontal Equations

These are incredibly simple, because the x-velocity doesn't change after the projectile has been projected.

$$\begin{cases} \Delta x = v_{x0}t \\ v_x = v_{x0} \end{cases}$$

4.3.2 Vertical Equations

Let g be $10m/s^2$ in the positive direction (hence the negative). Notice that these are just the constant acceleration equations, since the force of gravity is constant acceleration! In other words, $a_y = -g$.

$$\begin{cases} v_y = v_{y0} - gt \\ \Delta y = \frac{1}{2}(v_y + v_{y0})t \\ \Delta y = v_{y0}t - \frac{1}{2}gt^2 \\ v_y^2 = v_{y0}^2 - 2g\Delta y \end{cases}$$

Using the horizontal equation to resolve the time variable in the third vertical equation, we can derive a formula for y-position given the other factors:

$$\Delta y = v_{y0}\left(\frac{\Delta x}{v_{x0}}\right) - \frac{1}{2}g\left(\frac{\Delta x}{v_{x0}}\right)^2$$

If the origin is the release point, $\Delta x = x$ and $\Delta y = y$. This results in a quadratic, indicating that the path of a projectile ignoring air resistance models a parabola:

$$y = \left(-\frac{g}{2v_{x0}^2}\right)x^2 + \left(\frac{v_{y0}}{v_{x0}}\right)x$$

This is a pretty impractical equation for use, and you needn't memorize it. However, it serves nicely to emphasize that the path of an ideal projectile is a parabola.

4.4 Newton's Laws

Newton's three laws of mechanics, and the idea of force, are what underpin the entirety of classical mechanics.

4.4.1 The First Law

The first law states:

`An object at rest will remain at rest, or if it is in uniform motion, it will continue as such unless acted upon by a net force.`

This is called **inertia**. Larger (strictly, more massive) objects have more inertia. The first law dictates that **changes in velocity are caused by forces**; we know that change in velocity must mean acceleration, so forces must be tied to acceleration. Note that inertia is a property of matter, not a force itself.

4.4.2 The Second Law

$$\vec{F} = m\vec{a}$$

The second law relates the net force, the vector sum of all forces, to the acceleration of a system. Strictly, **the net force on an object/system is equal to the mass of the object/system multiplied by its acceleration**. The unit of measure used in the AP curriculum is the Newton, N; this is kilograms times acceleration, which is $\frac{kg*m}{s^2}$. Remember that since this is a vector equation, it in fact comprises *two* equations:

$$\begin{cases} {}_{net}F_x = ma_x \\ {}_{net}F_y = ma_y \end{cases}$$

Note that the m used in the second law equation and the m used in the definition of the newton are different: the former means mass, while the latter means meters.

4.4.3 The Third Law

Possibly the most famous of Newton's three laws. The third law dictates the interactions between two systems, stating that:

`When an interaction takes place between two systems, each system exerts a force on the other, and these two forces are equal in magnitude and opposite in direction.`

These forces are sometimes called a **action-reaction pair**. The third law is the backbone of most movement; it's how you walk, even!

4.5 Forces Overview

Forces can be divided into two types: **contact** forces, which involve two systems physically touching, and **noncontact** forces, which involve two systems not physically touching.

4.5.1 Normal Force

The first contact force is the **normal force** $\vec{N}$, the force between two surfaces that acts perpendicular to the surface; it's what allows an object to rest on another. For example, a ball resting on a table is pushing down on the table with its weight, and the table is pushing back up with an equivalent force; this is why the ball doesn't sink into the table!

4.5.2 Friction

The second contact force is the **friction force** $\vec{f}$, the force between two surfaces that acts parallel to the surface. The two types of friction are **static** ($\vec{f_s}$) and **kinetic** ($\vec{f_k}$); the former is when the two surfaces experience a force but don't actually move (e.g. rubbing rubber against rubber), while the latter is when two surfaces experience a force and slide over each other (e.g. rubbing metal on metal).

Kinetic Friction: Kinetic friction is produced any time **two surfaces are sliding over each other**. The amount of kinetic friction produced is directly correlated to the normal force, effectively how hard the surfaces are pressed into each other. Strictly, the magnitude of the kinetic friction is:

$$f_k = \mu_k N$$

where μ_k is the coefficient of friction between the two surfaces; think of this as a measurement of how slide-y they are against each other. N is the normal force, not a measure in Newtons!

Static Friction: Static friction is slightly different than kinetic friction. At an angle, an object at rest on a surface won't move until the surface reaches a certain angle; this is called the **angle of repose**, and this is caused by static friction. The magnitude of the maximum static friction is:

$$f_s \leq f_s^{max} = \mu_s N$$

where μ_s is the coefficient of static friction, generally greater than μ_k. N is again the normal force, not a measure in Newtons. f_s^{max} is only the maximum value for f_s, the actual static friction experienced by an object, which in turn is **only enough to offset the force in the other direction**. For example, if a box is being pushed with 15 N of force and $f_s^{max} = 30$ N, $f_s = 15$!

4.5.3 Tension and Compression

The third contact force, **tension**, is the pulling force exerted on an object by a rope or rod. Relatedly, compression is the pushing force exerted on an object by a rigid rod; ropes can't push. Ropes and rods are ideal and massless in AP Physics. Both tension and compression affect both involved objects equally, and are always directed *along* the rope or rod. The tension force is seen much more often than compression, and it's important to note that the tension force stays the same on a single rope across any pulleys it's attached to (pulleys are ideal, frictionless, and massless in AP Physics). In fact, just draw the free-body diagrams for each object (next section!) with the rope's angle at the attachment point to the object, and completely disregard any intermediate pulleys.

4.5.4 Spring Force

The last contact force is the **spring force**, the force exerted by a spring on the attached object. **Hooke's Law** defines the spring force as:

$$\vec{F} = -kx\hat{u}$$

where F is the force, k is the spring constant, x is the displacement, and $\hat{u}$ is the unit vector in the direction of the spring. The spring force is always directed opposite to the displacement of the object from its equilibrium position. You might see Hooke's Law without the $\hat{u}$, in which case it's assumed that the spring is in the x direction.

When calculating x, remember that the spring itself also has a "default", or **nominal**, length L_0; the total length L of a spring is equal to the nominal length plus the displacement, or $L = L_0 + x$, where positive x is extension and negative x is compression.

The **stress** and **strain** are also important spring-related concepts, although they aren't entirely required. Stress is the force applied per unit area, and strain is the ratio of the change in length ΔL to the original length L_0. These tie into **deformation**, of which there are two types: **elastic** deformation is when the object returns to its original shape after the force is removed, and **plastic** deformation is when the object does not return to its original shape after the force is removed. Plastic deformation occurs when excessive stress is applied to a material; the minimum level of stress required to plastically deform a material is that material's **elastic limit**.

4.5.5 Gravity

The only non-contact force in the AP Physics - Mechanics curriculum is gravity, the attraction between any two objects. (We'll be back to this later.) The gravitational attraction between the Earth and an object near it is called the object's **weight**, defined as $W = mg$ (mass * gravity).

4.5.6 Statics, Equilibrium, and Dynamics

Statics is the study of forces in equilibrium; in turn, **equilibrium** is when the net force on an object is zero. In the next section on free-body diagrams, you'll start with examples of statics. Recall Newton's second law, ${}_{net}F = ma$; in equilibrium, ${}_{net}F = 0$, meaning that $a = 0$ and *all forces in the x- and y-directions cancel each other out.* Statics are much simpler to solve because of this cancelling.

Dynamics is the study of forces in motion. In dynamics, ${}_{net}F \neq 0$, meaning that $a \neq 0$ and *not all forces in the x- and/or y-directions cancel each other out.* Dynamics are more complicated to solve because of this lack of cancelling, and you will need to account for the mass and acceleration of each object in the physical system being analyzed.

Do remember that the equation is ${}_{net}F = ma$, not ${}_{net}F = Wa$; mass (in kg) is not the same as weight (in N), although a weight in Newtons is sometimes provided! Be wary around the difference between mass and weight.

4.6 Free-Body Diagrams

A **free-body diagram** is a diagram of a single object in a physical system that depicts all force vectors acting upon it as arrows coming out of the object. A single system can have dozens of free-body diagrams drawn for it, as many as the number of objects in the system. Since these are obviously graphical representations, it's hard to put them into this document. See the textbook.

4.6.1 Analyzing Physical Systems

1. Draw a free-body diagram for each object.

2. Add and label arrows for all forces ON each object.

3. Choose axes such that the number of forces acting on each axis is maximized (this is usually the basic axes, or in an inclined plane problem, the axes of the plane).

4. Resolve forces not along the axes into their components along each axis.

5. Determine the sum of all force components on each object in each direction, resulting in ${}_{net}F_x$ and ${}_{net}F_y$ for each.

6. Apply the second law to equate the sum of force components to ma_x and ma_y.

7. Solve the resulting system of equations.

4.6.2 Inclines and Axes

In certain scenarios, the default horizontal and vertical axes are inefficient; the most common example of this is on an inclined plane. In this case, it makes sense to change x- and y-axes such that the x-axis is parallel to the incline and the y-axis perpendicular, so friction and acceleration have only x-components and the normal force only a y-component.

Any forces not acting parallel to an axis must be decomposed into their components along the axes. Weight is one such force; however, the decomposition is the same each time:

$$\begin{cases} W_x = mg\sin\theta \\ W_y = mg\cos\theta \end{cases}$$

4.6.3 Pulleys and Sign Conventions

Solving free-body diagram equations is confusing if you don't adopt consistent sign conventions. A **sign convention** means that every force in an arbitrary direction is marked positive, and forces in the other direction are marked negative. Since you should have broken down all forces into their components along the axes, there should only be 2 sets of positives and negatives. For example, with up being positive, a force of 5 N up added to a force of 2 N down is $5 + (-2) = 3$ N.

Selecting sign conventions is particularly important when working with pulleys, where *rotation* in a certain direction about the pulley must always be considered positive (standard is counterclockwise).

4.7 Forces in Free-Body Diagrams

4.7.1 Tension of Multiple Ropes

When examining an object with multiple tension forces, simply break up each tension force into its component vectors and solve. For example, for a box suspended by two ropes at angles of θ and ϕ degrees respectively, break each down into their component vectors, equate them to 0 (in equilibrium) or ma (not in equilibrium), and solve the system.

For an object suspended by multiple ropes in the same direction, the tension force is *equally split across each rope*, regardless of what it's attached to.

4.7.2 Tension of Accelerating Objects

For the simplest example of tension, a static object only supported by a rope, the tension force T only has to counterbalance the weight, such that $T = W = mg$. So what happens if the object is accelerating? Remember that $F = ma$; in the former example, $a = g$, so we could substitute it. However, when the object is accelerating, $a \neq g$; rather, $a = a_{net}$, where a_{net} is the net acceleration of the system. Therefore:

$$T = ma_{net}$$

If the object is affected by gravity (which it likely is), then $a_{net} = a_{ext} + g$, where a_{ext} is the external acceleration of the object in the direction of the rope. Therefore, the tension force is:

$$T = m(a_{ext} + g)$$

4.7.3 Gravity and Applied Weight

Solving the aforemetioned system of an object supported by a rope yields the equation $W = mg$: an object's weight is equivalent to its mass times the force of gravity.

In a vertically accelerating object, such as a person inside a moving elevator, the *applied* weight on the object differs from its weight in a static system. Specifically, the applied weight is:

$$w_{app} = w - ma = mg - ma = m(g - a)$$

For example, a person with weight 60 kg would have weight $W = mg = 60 * 10 = 600$ N in a static system, but if they accelerate at $5m/s^2$ upward, they would have applied weight $w_{app} = m(g-a) = 60(10 - (-5)) = 60 * 15 = 900$ N. The acceleration force is negative here because it acts in the opposite direction of gravity, which the sign convention here dictates is positive.

4.7.4 Depression of a Rope

I don't think this is on the AP curriculum, but I stumbled across this on my own. For a horizontal flexible rope with a force $F_\perp$ pushing down on the middle, the tension in the rope with angle of depression θ is:

$$T = \frac{F_\perp}{2\sin\theta}$$

4.8 Torque

Torque is a measure of the *rotational* force applied to an object. It is denoted with tau τ and defined as:

$$\tau = rF\sin\theta$$

where r is the distance from the pivot point to the point of force application, F is the force applied, and θ is the angle between the force and the lever arm. The lever arm is the shortest distance from the pivot point to the line of action of the force.

4.8.1 Balancing Torques and Rotational Equilibrium

In order for an object to be in rotational equilibrium, the sum of the torques on the object must be 0. This is because if the sum of the torques is not 0, then the object will rotate in the direction of the net torque. Therefore, the sum of the torques on an object is 0:

$$\sum \tau = 0$$

This is analogous to the sum of the forces on an object being 0 for translational equilibrium.

To balance torques on an object, take a pivot point, and sum the torques on the object about that pivot point. For example, if a rod is suspended by a rope at its center, the torques on the rod are balanced about the pivot point of the rope, since the torques on either side of the pivot point are equal and opposite. Then, solve for the unknown variable.

4.8.2 Center of Mass

The **center of mass** of an object is the point at which the object's mass is concentrated. For a uniform object, the center of mass is the geometric center of the object. For a non-uniform object, the center of mass is the weighted average of the object's mass distribution. For example, the center of mass of a meter stick is at its center, but the center of mass of a meter stick with a weight on one end is closer to the end with the weight. The formula of this is:

$$x_{cm} = \frac{\sum mx}{\sum m}$$

where x_{cm} is the center of mass, m is the mass of a certain part of the object, and x is the distance of that part from the origin. Torques act about the center of mass of an object.

4.9 Uniform Circular Motion

Uniform circular motion is the motion of an object in a circle at constant speed. Since the velocity vector does not stay constant, the object is constantly accelerating, but the object's *speed* remains constant - this is because the acceleration is purely inwards, to maintain a circular trajectory. This *centripetal* acceleration of an object in uniform circular motion is:

$$a_c = \frac{v^2}{r}$$

where v is the speed of the object and r is the radius of the circle. The centripetal acceleration is always directed towards the center of the circle. The centripetal force is the force that causes this acceleration, and is given by:

$$F_c = ma_c = m\frac{v^2}{r}$$

4.9.1 Period and Frequency

The **period** of an object in uniform circular motion is the time it takes for the object to complete one full revolution, denoted as T. The **frequency** of an object in uniform circular motion is the number of revolutions the object completes per second. The period and frequency are related by:

$$f = \frac{1}{T}$$

The units of the period are seconds, and the frequency is measured in Hertz (Hz), or $\frac{1}{s}$.

Speed and period can be related as follows:

$$v = \frac{2\pi r}{T}$$

4.9.2 Angular Quantities

Angular quantities are quantities that describe the motion of an object in a circle. The **angular displacement** θ is the angle through which an object moves, measured in radians. The **angular velocity** ω is the rate of change of angular displacement, measured in radians per second, and calculated as $\omega = \frac{2\pi}{T}$. The angular velocity and angular acceleration are related to the linear velocity and linear acceleration by:

$$v = r\omega$$

4.9.3 Banked Turns

A **banked turn** is a turn in which the road is tilted at an angle θ from the horizontal. The normal force is split into two components: one perpendicular to the road, and one parallel to the road. The perpendicular component is equal to the weight of the car, and the parallel component is equal to the centripetal force. The normal force is given by:

$$N = mg\cos\theta$$

The centripetal force is given by:

$$F_c = mg\sin\theta$$

4.10 Gravity

Newton's universal law of gravitation states that every object in the universe exerts a gravitational force on every other object. The magnitude of this force is given by:

$$F_g = G\frac{m_1 m_2}{r^2}$$

where G is the universal gravitational constant, m_1 and m_2 are the masses of the two objects, and r is the distance between the two objects. The direction of the force is always towards the other object. The gravitational constant is given by:

$$G = 6.67 \times 10^{-11}\frac{N \cdot m^2}{kg^2}$$

You can solve a lot of problems of orbital motion by relating centripetal force to the magnitude of the force of gravity and solving for the unknown:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

For instance, for satellite orbits, when one mass is much larger than the other, the smaller mass orbits the larger mass in a circle. Manipulating the above, the centripetal force is provided by the gravitational force, and the speed of the satellite is given by:

$$v = \sqrt{\frac{GM}{r}}$$

It may also be sometimes necessary to replace v with $\frac{2\pi r}{T}$. This yields:

$$\frac{4\pi^2 r^3}{T^2} = GM$$

This can then be manipulated as needed.

Manipulating the universal law of gravitation also gives the force of gravity on a planet:

$$g = G\frac{M}{r^2}$$

4.11 Work and Energy

Work is the product of the magnitude of the displacement and the component of the force parallel to the displacement. It is given by:

$$W = Fd\cos\theta$$

where F is the force applied, d is the displacement, and θ is the angle between the force and the displacement. The units of work are Joules (J), or $N \cdot m$. Work is a scalar quantity.

The net work done on a system is given by:

$$W_{net} = \Delta KE = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

where KE is the kinetic energy of the system, m is the mass of the system, and v_f and v_i are the final and initial velocities of the system, respectively. The net work done on a system is equal to the change in kinetic energy of the system. The kinetic energy - that is, energy of motion of an object - is given by:

$$KE = \frac{1}{2}mv^2$$

4.11.1 Conservative Forces

A **conservative force** is a force that does the same work on an object regardless of the path taken. Gravity and spring forces are examples of conservative forces. The work done by a conservative force is given by:

$$W = -\Delta PE$$

where PE is the potential energy of the system. The potential energy of a system is the energy the system will release when a force stops acting on it, such as the restoring forces of gravity or a spring. The graviational and elastic potential energies are given by:

$$GPE = mgh$$

$$EPE = \frac{1}{2}kx^2$$

where m is the mass of the object, g is the acceleration due to gravity, h is the height of the object, k is the spring constant, and x is the displacement of the spring from its equilibrium position.

4.11.2 Nonconservative Forces

A **nonconservative force** is a force that does not do the same work on an object regardless of the path taken. Friction is an example of a nonconservative force. The work done by a nonconservative force is given by:

$$W = -\Delta PE + \Delta KE$$

where PE is the potential energy of the system and KE is the kinetic energy of the system. Examples of nonconservative forces include friction, in which the energy is ”lost” to heat, sound, and other forms of energy.

4.11.3 Conservation of Energy

The total energy of a system is the sum of the kinetic and potential energies of the system. The total energy of a system is conserved under the action of a conservative force, meaning that it remains constant. This is known as the **law of conservation of energy**. The total energy of a system is given by:

$$E = KE + PE$$

When a conservative force acts on this system, KE can be changed into PE or vice versa, but the total energy of the system remains constant. This can be written as:

$$E_i = E_f$$

where E_i is the initial energy of the system and E_f is the final energy of the system. In terms of the change in KE and PE:

$$\Delta KE + \Delta PE = 0$$

4.11.4 Power

Power is the rate at which work is done. It is given by:

$$P = \frac{W}{t}$$

where W is the work done and t is the time taken. The units of power are Watts (W), or J/s.

4.12 Momentum

Momentum is the product of the mass and velocity of an object. It is given by:

$$p = mv$$

In fact, Newton's second law was originally written in terms of momentum:

$$F = \frac{\Delta p}{\Delta t}$$

4.12.1 Conservation of Momentum

Like energy, momentum is a conserved quantity. The **law of conservation of momentum** states that the total momentum of a system is constant. This can be written as:

$$p_i = p_f$$

where p_i is the initial momentum of the system and p_f is the final momentum of the system. In terms of the change in momentum:

$$\Delta p = 0$$

4.12.2 Collisions

During a collision between two objects, momentum is conserved in each direction: the total momentum in the x-direction is conserved, and the total momentum in the y-direction is conserved. This can be written as:

$$p_{i,x} = p_{f,x}$$

$$p_{i,y} = p_{f,y}$$

In general, for two objects colliding in one dimension (and the same can be extended to each dimension of a 2-dimensional problem):

$$m_1 v_{1,i} + m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f}$$

Collisions can also be *elastic* or *inelastic*. Elastic collisions are those in which kinetic energy is conserved, while inelastic collisions are those in which kinetic energy is not conserved. In elastic collisions, you can also apply conservation of energy to solve the system.

4.12.3 Impulse

Impulse is the change in momentum of an object. It is given by:

$$J = \Delta p = F\Delta t$$

where F is the force applied and Δt is the time taken. Impulse is also equal to the area under a force-time graph, for the same reason.

4.13 Simple Harmonic Motion

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4.14 Final Equations

4.14.1 1D Kinematics

$$\begin{cases} v = v_0 + at \\ \Delta x = \frac{1}{2}(v + v_0)t \\ \Delta x = v_0 t + \frac{1}{2}at^2 \\ v^2 = v_0^2 + 2a\Delta x \end{cases}$$

4.14.2 Free Fall

$$\begin{cases} t_{peak} = \frac{v_0}{g} \\ h = \frac{v_{y0}^2}{2g} \\ v_y = v_{y0} - gt \\ \Delta y = \frac{1}{2}gt^2 \end{cases}$$

4.14.3 Newton's Laws

$$\begin{cases} F = ma \\ W = mg \\ W_{eff} = mg - ma \\ F_{f,s} \leq \mu F_N \\ F_{f,k} = \mu F_N \\ T = ma_{net} \end{cases}$$

4.14.4 Equilibrium

$$\begin{cases} W_y = mg\cos\theta \\ W_x = mg\sin\theta \\ \sum F = 0 \\ \sum \tau = 0 \end{cases}$$

4.14.5 Uniform Circular Motion

$$\begin{cases} \omega = \frac{2\pi}{T} \\ v = r\omega \\ a_c = \frac{v^2}{r} = r\omega^2 \\ F_c = ma_c = \frac{mv^2}{r} \\ f = \frac{1}{T} \\ T = \frac{2\pi r}{v} \end{cases}$$

4.14.6 Banked Turns

$$\begin{cases} \tan\theta = \frac{v^2}{rg} \\ N = mg\cos\theta \\ F_c = mg\sin\theta \end{cases}$$

4.14.7 Gravity

$$\begin{cases} G = 6.67 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2} \\ F_g = \frac{Gm_1m_2}{r^2} \\ v = \sqrt{\frac{Gm}{r}} \\ g = \frac{GM}{r^2} \\ GM = \frac{4\pi^2 r^3}{T^2} \end{cases}$$

4.14.8 Work and Energy

$$\begin{cases} W = Fd\cos\theta \\ W_{net} = \Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \\ GPE = mgh \\ EPE = \frac{1}{2}kx^2 \\ KE = \frac{1}{2}mv^2 \\ KE_i + PE_i = KE_f + PE_f \\ P = \frac{W}{\Delta t} \end{cases}$$

4.14.9 Momentum and Impulse

$$\begin{cases} p = mv \\ p_{i,x} = p_{f,x} \\ p_{i,y} = p_{f,y} \\ I = \Delta p = F\Delta t \end{cases}$$

4.14.10 Simple Harmonic Motion

$$\begin{cases} F = -kx \\ T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}} \\ x = A\cos(\omega t) \\ x_{max} = A \\ v = -v_{max}\sin(\omega t) \\ v_{max} = A\omega \\ a = -a_{max}\cos(\omega t) = -\omega^2 x \\ a_{max} = A\omega^2 \end{cases}$$

Part III

Computer Science

5 AP Computer Science A

5.1 Polymorphism

This is something I see a lot of people struggle on, so I'll write out a handy chart here. This is likely one of the only things from the APCSA curriculum that I'll put in this article; the rest is simple enough that I'll tell you to just go Google it or something. Basic class structure, inheritance, abstraction, all that you should just ask a teacher or consult the Internet.

Let subclass `P` inherit class `O`. On an object of type `P` declared as type `O`, like so:

```
O thisObj = new P();
```

Calling `thisObj.N()` for some method `N()` yields the following results, depending on which class(es) implement it:

	`O` implements `N()`	`O` does not implement `N()`
`P` implements `N()`	Runs `N()` from `P`	Fails*
`P` does not implement `N()`	Runs `N()` from `O`	Fails (obviously)

*Unless `thisObj` is typecast to type `P` before calling, like so:

```
((P) thisObj).N();
```

This is a concept known as **overriding** (not to be confused with *overloading*, when multiple functions are defined with the same name but different signatures). What happens is that at compile-time, Java only "adds" the functions defined in the class the object is **declared** as, but if one of those functions has a different method body in the class the object is **instantiated** as, that method body **overrides** the method body in the declared class and is run whenever that function is called.